Reply to "Comment on 'Multiple scattering: The key to unravel the subwavelength world from the far-field pattern of a scattered wave' "

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In a recent paper [F. Simonetti, Phys. Rev. E **73**, 036619 (2006)], it is suggested that multiple scattering is the key to achieve subwavelength resolution imaging from far-field measurements. In support of this claim, the same paper also reports on an experiment in which a resolution better $\lambda/3$ is achieved, λ being the wavelength of the probing wave. In a Comment [J. de Rosny and C. Prada, preceding paper, Phys. Rev. E. **75**, 048601 (2007)], this argument is disputed and it is claimed that subwavelength resolution is possible under the Born approximation. However, in de Rosny and Prada the effect of measurement noise, which is central to the use of subwavelength resolution techniques in practice, is not considered. By means of an example similar to that discussed in de Rosny and Prada, this paper confirms that multiple scattering is indeed a "*key*" factor to achieve subwavelength resolution with real, noisy measurements.

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I appreciate the interest of the authors in my paper [1]; their Comment [2] suggests that it will be useful to expand on a number of points, particularly the influence of measurement noise. First, for clarity, it is crucial to agree what it is meant by the terms imaging and detection. Here and in [1], it is assumed that imagining includes tomography, which provides the spatial distribution of the physical properties of an object, and shape reconstruction which describes the geometry of an object in space. The term *subwavelength detection* used in [2] might lead to confusion. As an example, in acoustics it is well known that a bubble can produce a large echo even when the wavelength, λ, of the incident wave is several orders of magnitude larger than the size of the bubble. In this context, the detection of a subwavelength scatterer does not pose any difficulty. On the other hand, if the position of two neighboring bubbles or even their shape have to be determined, then the resolution problem occurs. For this reason, the term imaging rather than detection is used.

Reference [1] contains two fundamental statements. The first is that while under the Born approximation (BA) the far-field measurements only depend on the spatial frequencies of the object lower than $2k$ [3] ($k=2\pi/\lambda$), in the presence of multiple scattering (MS) the measurements depend on the entire spectrum of the object function including the spatial frequencies higher than $2k$. The second statement is based on the observation that since the subwavelength structure of the object affects the far-field measurements, it is possible to image the object with subwavelength resolution. This result is confirmed by numerical and experimental evidence reported by other authors independently (see Refs. [17–19] in [1] and a paper by Belkebir *et al.* [4]).

In [2] the authors state that "Contrary to the conclusion of Simonetti [1], we maintain that MS is not the key for subwavelength detection. Indeed even with no MS between subwavelength structures, subwavelength detection is still possible." I agree that subwavelength resolution under the BA is possible provided that the scattering mechanism is actually described by the BA and that the noise level is low. Indeed, in Ref. [1] it is clearly stated that unlimited resolution is possible under the BA, quoting Sec. IV A: "Under the Born approximation there are two approaches to super resolution. The first is based on the analyticity of the function $\tilde{O}(\Omega)$ when the support D is finite. In this case, $\tilde{O}(\Omega)$ can be extrapolated to the exterior of the limiting Ewald's sphere by analytic continuation [30] so achieving unlimited resolution." From a different perspective, unlimited resolution is a consequence of the uniqueness of the solution to the inverse scattering problem. However, in the presence of noise this statement alone is insufficient; again quoting Sec. IV A "However, as observed by several authors [31–35], analytic continuation is not practically feasible due to its severe instability and high sensitivity to noise." The noise level varies depending on the type of application and whether electromagnetic or ultrasonic waves are used. As an example, with current ultrasound technology the achievable signal-to-noise (SNR) ratio is well below 50 dB, and in the experiment reported in [1] it was below 10 dB. Therefore, the study of subwavelength resolution imaging cannot be separated from noise considerations. This is the major point of disagreement with the claims made in [2]. The aim of this paper is to confirm that MS is crucial to achieve subwavelength resolution due to the presence of noise in real measurements.

First, it is important to observe that under the BA the feasibility of subwavelength resolution is compromised by the presence of noise, because of the severe instability of the inverse scattering problem which causes the solution not to vary continuously with the measurements. As a result, even small measurement errors can lead to significant artifacts in the reconstructed image. The severe instability is due to the fact that under the BA, there exists a one-to-one mapping between the far-field measurements and the spatial frequencies of the object which are lower than $2k$ and thus, there is no link between the measurements and the frequencies higher than $2k$, i.e., the very fine structure of the object. Currently available imaging technology addresses the instability by assuming that the object function does not contain frequencies higher than $2k$ [see Eq. (14) in [1]], leading to

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standard diffraction tomography. This is a type of regularization and poses a practical lower bound to the resolution of $\lambda/2$ [3]. For completeness, it should be mentioned that if prior knowledge about the object is available, this can partially compensate for the missing higher spatial frequencies and can lead to subwavelength resolution also under the BA. However, this is not the case considered in [1] where even the factorization method does not require any prior knowledge about the shape, consistency, size, and number of the objects to be imaged. Note that this has been confirmed experimentally in the case of ultrasound probing in [5].

On the other hand, what it is shown in [1] is that when MS is considered, a single far-field measurement depends on the entire spectrum of the object. This observation is crucial to achieve subwavelength resolution since the missing physical link between the measurements and the very fine structure of the object is now restored. As a result, in the context of noisy measurements which are representative of any real experiment, [1] claims that MS is the *key* to subwavelength resolution imaging.

The analysis performed in [2] does not consider noise, and therefore does not account for the instability of the inverse problem. In order to demonstrate that MS is crucial to achieve subwavelength resolution, the example of two point scatterers discussed in Ref. [2] is revisited. First it is important to clarify that the experiment reported in Ref. [1] cannot be described by two point scatterers since the diameter of the rods is $\lambda/5$ which is not negligible compared to the wavelength and to the relative distance between the rods ($\lambda/3$). Therefore, the example provided in Ref. [2] is not representative of the experiments in Ref. [1]. In addition, point scatterers are not suitable to study interactions at subwavelength scale, as they require the size of the scatterers to be much smaller than their relative distance which is itself smaller than the wavelength. This makes the experimental validation of such a model very challenging. Moreover, there are several errors in the definition of the parameters of the **S** matrix in Eq. (1) of [2]; two of them are discussed here. First, [2] assumes that the scattering coefficient for each point scatterer is unity. This assumption violates energy conservation since according to the optical theorem the excitation cross section would vanish. The second error is in the expression of the off-diagonal terms of the **S** matrix in the presence of MS. In particular, the authors assume that the field scattered by one point and reaching the other can be represented by a plane wave. This is incorrect because each scatterer is in the near field of the other. Note that the near-field interaction is the mechanism which encodes subwavelength information in the far field as discussed in [1]. Therefore, for a two-dimensional problem the higher-order scattering has to be modeled by using the exact Green's function, which is proportional to the zero order Hankel function of the first kind. In the absence of noise, the coefficients of **S** do not affect the resolution since the eigenfunctions of the far-field operator T_∞ do not depend on them. However, the nonzero eigenvalues of the operator are dependent on the coefficients of **S**, and influence the accuracy with which the eigenfunctions can be estimated from a noisy T_∞.

In this paper, the MS phenomenon is described according to the Foldy-Lax method [6], which satisfies energy conservation, is self-consistent, and avoids the convergence problems which can occur using the Born series [3]. Moreover, it is assumed that the scattering is elastic; in this case by energy conservation it is easy to show (see, for instance, [7]) that the scattering coefficient, o, of a pointlike scatterer has to have the form

$$o = -2[e^{ip} + i], \tag{1}$$

where p depends on the scatterer geometrical and material properties. In general p is a function of frequency so as to account for the dispersion of the scattering coefficient and its resonances. The motivation for a particular choice of the dispersion characteristic of p goes beyond the scope of this paper, therefore it is simply assumed that p is constant and equal to $3\pi/4$.

It is then a simple matter to calculate the discretized version of the far-field operator under the BA, T_{BA}, and with the Foldy-Lax method, T_{MS}. As correctly shown in [2], in the absence of noise the factorization method (FM) leads to unlimited resolution with or without MS. However, when noise is present the resolution deteriorates and the FM applied to the T_{MS} leads to a much higher resolution than when it is applied to T_{BA}. To demonstrate this, noise is added to the exact T matrices,

$$T_n = T + \mathcal{N}, \tag{2}$$

where $\mathcal{N}_{ij}$ is a complex number with a random Gaussian amplitude with standard deviation s and random phase uniformly distributed in $[-\pi \quad \pi]$; it is assumed that the random process is stationary for different scattering experiments. The noise level, n, is estimated from

$$n = \frac{sN}{\|T\|}, \tag{3}$$

where $\|\cdot\|$ refers to the Frobenius norm and N is the dimension of T. In the calculations, the same realization of noise was added to T_{BA} and T_{MS}. However, since $\|T_{BA}\| \neq \|T_{MS}\|$, a scaling factor was applied to $\mathcal{N}$ so as to ensure that n was the same in both cases. Moreover, it is assumed that the scatterers are probed with 57 point transducers arranged as in the experiment reported in [1]. However, while the distance of the scatterers from the array is the same as in the experiment, the more challenging case in which the scatterers are $\lambda/10$ apart is considered.

Figure 1(a) shows the pseudospectrum [Eq. (35) in [1]] when the FM is applied to T_{BA} with five different noise levels: 0.01, 0.1, 0.2, 0.3, and 0.4 %. It is clear that when the noise level is low the FM applied to T_{BA} can resolve the scatterers; however, when the noise is as low as 0.4%, the FM fails to resolve them. On the other hand, Fig. 1(b) shows the pseudospectra obtained when the FM is applied to T_{MS} and the same noise levels as in Fig. 1(a) are added. Thanks to MS, the FM can now resolve the scatterers with noise levels in excess of 4% as shown in Fig. 2, thus ten times larger than the corresponding critical noise level under the BA. A similar result has been obtained by Marengo and Gruber [8] in the context of the time reversal and Multiple Signal Classification method.

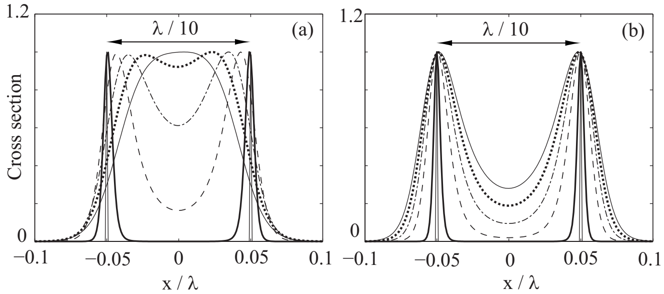

FIG. 1. Cross section of the factorization method pseudospectrum. The scatterers are located at $x/\lambda = \pm 0.05$. The scattered field is calculated: (a) under the BA; (b) considering MS. The exact T matrices are corrupted with five different levels of additive noise: (thick solid curve) 0.01%; (dashed curve) 0.1%; (dash dot curve) 0.2%; (dotted curve) 0.3%; (thin solid curve) 0.4%.

In order to gain an insight into the mechanism which leads to the resolution enhancement by MS, Fig. 3 shows the angular distributions of the modulus of the scattering amplitude when the incident field is perpendicular, Fig. 3(a), and parallel, Fig. 3(b), to the scatterers. The pattern of the scattered field is predicted with the Foldy-Lax model (solid curves) and under the BA (dashed curves). The latter approaches the pattern of a monopole source radiating from a single point between the scatterers and is only slightly affected by the illumination direction. On the other hand, when MS is considered, the shape of the far-field pattern is very sensitive to the illumination direction which leads to a pattern resembling the combination of a monopole and a dipole when the illumination is parallel to the scatterers, Fig. 3(b). Note that this has also been observed experimentally [9]. Thus, in the absence of MS, in order to appreciate the difference between the field scattered by a single scatterer or two scatterers, the noise level has to be low (the dashed curves in Fig. 3 are close to a circle). On the other hand, a larger noise level can be tolerated when MS occurs, thanks to the pronounced angular diversity of the far-field patterns. This angular diversity is a manifestation of the encoding of subwavelength information in the far field and is crucial to resolve the scatterers in the presence of noise.

To take full advantage of the encoding mechanism the

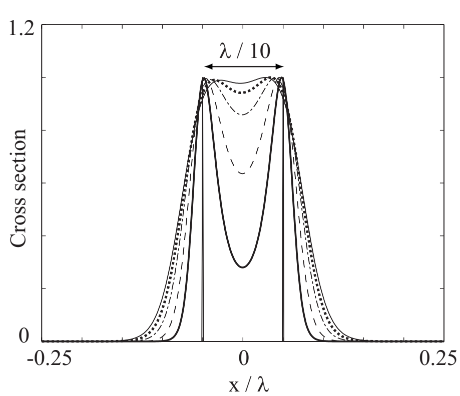

FIG. 2. Equivalent to Fig. 1(b) for larger noise levels of (thick-solid curve) 0.4%; (dashed curve) 1%; (dash dot curve) 2%; (dotted curve) 3%; (thin solid curve) 4%.

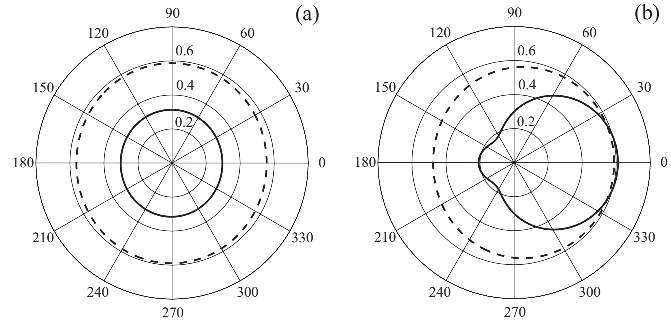

FIG. 3. Polar diagrams of the modulus of the scattering amplitude as a function of the observation direction; 0 degrees corresponds to the direction parallel to the scatterers. The scatterers are $\lambda/10$ apart. The diagrams are obtained for two illumination directions: (a) parallel to the scatterers; (b) orthogonal. The scattering amplitude is calculated with the BA (dashed curves) and with the Foldy-Lax model (thick solid curve).

scattering amplitude should be measured for all the possible illumination directions. Figure 4 shows the pseudospectrum of the FM when 57 transducers surround the scatterers at equal angular intervals and are placed in the far field. Under the BA, the scatterers can now be resolved with noise levels below 3%. On the other hand, thanks to MS the FM can resolve the scatterers also in the presence of noise as large as 30%. This is a very remarkable result and is the reason why in Ref. [1] MS is regarded as the "*key*" factor to achieve subwavelength resolution imaging.

Before concluding a few more issues need to be clarified. Quoting [2] "the expression of the Picard's theorem, at least as it is explained in the paper, remains exactly the same whether the MS is considered or not." This is correct and Ref. [1] did not claim the contrary, indeed a reference to a paper by Kirsch which discusses the theory of the FM with and without MS for point scatterers and extended objects was given (Ref. [68] in [1]). However, it has to be stressed that in the general case of extended objects, although the form of expression (35) in [1] remains the same whether or not MS is considered, the eigenvalues, μ, used in it change since the far-field operator is affected by the presence of MS.

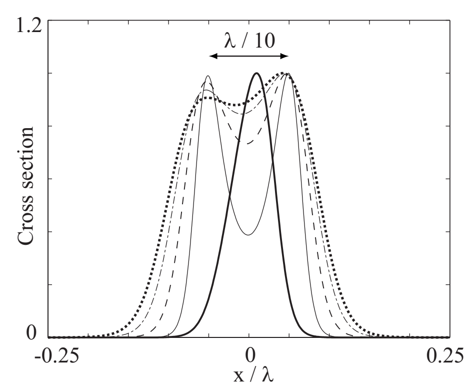

FIG. 4. Cross section of the FM pseudospectrum obtained with full view probing for different levels of noise: (thick-solid curve) BA with 3%; (thin solid curve) MS with 3%; (dashed curve) MS with 10%; (dash dot curve) MS with 20%; (dotted curve) MS with 30%.

In fact, if MS is present and no energy dissipation occurs, the optical theorem holds and it can be shown that the eigenvalues of T_∞ must lie on a circle of the complex plane passing through the origin with radius $\sqrt{2\pi/k}$ and centered along the line $\mathrm{Im}\{\mu\}=-\mathrm{Re}\{\mu\}$, with $\mathrm{Im}\{\mu\}>0$ [10]. In contrast, under the BA the optical theorem does not hold [11] and the eigenvalues are scattered over the entire complex plane. In the particular case of point scatterers, only the nonzero eigenvalues are sensitive to MS. Since only the zero eigenvalues contribute to Eq. (35) in [1] the authors of [2] conclude that super resolution is not related to MS. However, they do not consider that the eigenfunctions and eigenvalues in Eq. (35) have to be calculated from the far-field operator which contains noise. Due to noise, none of the eigenvalues vanishes and also the eigenfunctions are modified. The extent to which noise alters the eigenvalues and eigenfunctions is different depending on whether they are calculated from T_{BA} or T_{MS}, which as shown in Fig. 3 can have a very different structure. As confirmed by the numerical examples reported in this paper, the richer angular diversity of T_{MS}, which is due to the additional information encoded by MS, leads to an estimate of the eigenfunctions and eigenvalues which is more robust against noise than under the BA.

In [2], the authors state that the invariance of condition (35) in [1] with respect to MS contradicts the statement of Ref. [1]: "This confirms the argument discussed in the previous section and demonstrates how abandoning the Born approximation and including multiple scattering leads to super resolution." Expression (35) in [1] is consistent with the statement that MS leads to super resolution. However, it is also true that Eq. (35) itself does not demonstrate that the resolution with MS would be better than without it. This is because noise does not appear explicitly in Eq. (35). On the other hand, this conclusion can be drawn by observing that with MS, the scattering amplitude does contain information about the subwavelength structure of the object and this additional information, which is not present under the BA, is correctly interpreted by Eq. (35). This was consistent with the experiment reported in [1] in which MS is present.

Finally, it has to be observed that a super-resolution inversion algorithm based on the BA would have limited resolution even if it were applied to the exact scattered field calculated by solving the Helmholtz equation. In fact, such an algorithm would attempt to find the best fit between a forward model, based on the BA, and the input scattered field. If the scattering phenomenon were actually described by the BA, the method would lead to unlimited resolution. On the other hand, the scattered field obtained by solving the Helmholtz equation is different from that obtained under the BA due to MS effects. Therefore, the difference between the actual scattering pattern and the one predicted by the BA represents a form of coherent noise. With reference to Fig. 3 an inversion algorithm based on the BA leads to an unlimited resolution when it is applied to the data corresponding to the dashed curve but it would lead to a poor resolution when applied to the data associated with the solid curve.

In conclusion, my major disagreement with the argument of [2] is that the effect of noise is not considered in [2]. I agree that in the absence of noise (or at very small levels) subwavelength resolution can be achieved under the BA provided that the scattering mechanisms can actually be described by the BA. However, in this paper, by means of an example similar to that provided in [2], it is confirmed that MS is indeed a "*key*" factor to achieve subwavelength resolution since real measurements always contain noise.

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